Solution to Homework 2, MMAT5000

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1. **Solution:**

(a) For any $\varepsilon > 0$, $\exists\ \delta = \min\{\frac{\varepsilon}{16}, 1\}$ such that if

$$\|(x, y) - (1, -4)\| = \sqrt{(x-1)^2 + (y+4)^2} < \delta,$$

we have

$$\begin{aligned} |x^2y + 4| &= |x^2y - y + y + 4| \\ &\leq |x^2y - y| + |y + 4| \\ &\leq |x+1| \cdot |y| \cdot |x-1| + |y+4| \end{aligned}$$

Notice that

$$|x-1| \leq \sqrt{(x-1)^2 + (y+4)^2} < \delta,$$
$$|y+4| \leq \sqrt{(x-1)^2 + (y+4)^2} < \delta,$$
$$|x+1| \leq |x-1| + 2 \leq \delta + 2,$$
$$|y| \leq |y+4| + 4 \leq \delta + 4.$$

Then

$$|x^2y + 4| \leq (\delta+2)(\delta+4)\delta + \delta = (\delta^2 + 6\delta + 9)\delta \leq 16\delta \leq \varepsilon.$$

Therefore

$$\lim_{(x,y)\to(1,-4)} x^2y = -4.$$

(b) **Claim:** $\left|\dfrac{\sin t}{t} - 1\right| \leq |t|$, for all $t \in \mathbb{R} - \{0\}$.

Proof of Claim: Let $f(t) = \sin t - t + t^2$, $t \in [0, +\infty)$.

$$f'(t) = \cos t - 1 + 2t, \quad f'(0) = 0,$$

and

$$f''(t) = -\sin t + 2 > 0.$$

So $f'(t) > 0$ in $(0, +\infty)$ and hence $f(x) \geq 0$ on $[0, +\infty)$ due to $f(0) = 0$.
Therefore, $t - t^2 \leq \sin t$, $t \in (0, +\infty)$.
Similarly, let $g(t) = \sin t - t - t^2$, $t \in [0, +\infty)$.

$$g'(t) = \cos t - 1 - 2t, \quad g'(0) = 0,$$

and

$$g''(t) = -\sin t - 2 < 0.$$

So $g'(t) < 0$ in $(0, +\infty)$ and hence $g(x) \leq 0$ on $[0, +\infty)$ due to $g(0) = 0$. Therefore, $\sin t \leq t + t^2,\ t \in (0, +\infty)$.

$$t - t^2 \leq \sin t \leq t + t^2,\ t \in (0, +\infty),$$

$$-t \leq \frac{\sin t}{t} - 1 \leq t,\ t \in (0, +\infty),$$

and hence

$$\left| \frac{\sin t}{t} - 1 \right| \leq |t|,\ t \in \mathbb{R} - \{0\}.$$

For any $\varepsilon > 0,\ \exists\ \delta = \min\{\frac{\varepsilon}{23}, 1\}$ such that if

$$\|(x, y) - (-2, 3)\| = \sqrt{(x+2)^2 + (y-3)^2} < \delta,$$

$$\begin{aligned}
\left| \frac{\sin(9x + 2y^2)}{9x + 2y^2} - 1 \right| &\leq |9x + 2y^2| \\
&= |9(x+2) + 2(y-3)^2 + 12(y-3)| \\
&\leq 9|x+2| + 2|y-3|^2 + 12|y-3| \\
&\leq 9\delta + 2\delta^2 + 12\delta \\
&= \delta(2\delta + 21) \\
&\leq 23\delta \\
&\leq \varepsilon.
\end{aligned}$$

Here we used the fact that

$$|x+2| \leq \sqrt{(x+2)^2 + (y-3)^2} < \delta,$$

$$|y-3| \leq \sqrt{(x+2)^2 + (y-3)^2} < \delta.$$

Therefore

$$\lim_{(x,y)\to(-2,3)} \frac{\sin(9x + 2y^2)}{9x + 2y^2} = 1.$$

2. **Solution:**

(a) If we take the limits through the path $x = 0$,

$$\lim_{y\to 0} g(0, y) = \lim_{y\to 0} \frac{0 + y^2}{0 + 3y^2} = \frac{1}{3};$$

While taking the limits through the path $y = 0$ leads to

$$\lim_{x\to 0} g(x, 0) = \lim_{x\to 0} \frac{x^3 + 0}{2x^4 + 0} = \lim_{x\to 0} \frac{1}{2x} \to \begin{cases} +\infty, & x \to 0^+, \\ -\infty, & x \to 0^-. \end{cases}$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} g(x, y)$ does not exist.

(b) If we take the limits through the path $x = y^5 - y^2$,

$$\lim_{y\to 0} h(y^5 - y^2, y) = \lim_{y\to 0} \frac{(y^5 - y^2)^2 y}{y^5 - y^2 + y^2} = \lim_{y\to 0}(1 + y^6 - 2y^3) = 1;$$

While taking the limits through the path $x = 2y^5 - y^2$ leads to

$$\lim_{y\to 0} h(2y^5 - y^2, y) = \lim_{y\to 0} \frac{(2y^5 - y^2)^2 y}{2y^5 - y^2 + y^2} = \lim_{y\to 0}(\frac{1}{2} + 2y^6 - 2y^3) = \frac{1}{2}.$$

Therefore, $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist.

Remark: However, we have

$$\lim_{(x,y)\to(0^+,0)} h(x, y) = 0.$$

We will show it by $\varepsilon - \delta$ definition: For any $\varepsilon > 0$, $\exists\ \delta = \sqrt{\varepsilon} > 0$ such that if

$$x > 0, \quad \|(x, y) - (0, 0)\| = \sqrt{(x)^2 + (y)^2} < \delta,$$

$$\begin{aligned}
\left|\frac{x^2 y}{x + y^2} - 0\right| &\le \left|\frac{x^2 y}{x}\right| \\
&= |xy| \\
&\le |x| \cdot |y| \\
&\le \delta^2 \\
&\le \varepsilon.
\end{aligned}$$

Here we used the fact that

$$|x| \le \sqrt{(x)^2 + (y)^2} < \delta,$$

$$|y| \le \sqrt{(x)^2 + (y)^2} < \delta.$$

Therefore

$$\lim_{(x,y)\to(0^+,0)} h(x, y) = 0.$$

3. **Solution:** For each $(x, y) \in U$, we have

$$3x^2 + 5y^2 > 7.$$

Set $A(x, y) \overset{def}{=} 3x^2 + 5y^2 - 7 > 0$.
For any $(s, t) \in B_r((x, y))$, we have

$$\sqrt{(s - x)^2 + (t - y)^2} < r,$$

$$|s - x| < r, \ \ |t - y| < r,$$

$$|s + x| \le |s - x| + |2x| \le r + 2|x|,$$

$$|t + y| \le |t - y| + |2y| \le r + 2|y|.$$

Moreover,

$$\begin{aligned}3s^2+5t^2-7 &= 3(s^2-x^2)+5(t^2-y^2)+3x^2+5y^2-7\\ &= 3(s+x)(s-x)+5(t+y)(t-y)+A(x,y)\\ &\geq -3|s+x|\cdot|s-x|-5|t+y|\cdot|t-y|+A(x,y)\\ &\geq -3(r+2|x|)r-5(r+2|y|)r+A(x,y)\\ &= -8r^2-(6|x|+10|y|)r+A(x,y)\end{aligned}$$

Notice that

$$-8r^2-(6|x|+10|y|)r+A(x,y)>0$$

provided

$$r\in\left(0,\frac{\sqrt{(3|x|+5|y|)^2+8A(x,y)}-(3|x|+5|y|)}{8}\right).$$

Therefore, for each $(x,y)\in U$, the ball $B_r((x,y))\subset U$ provided

$$r\in\left(0,\frac{\sqrt{(3|x|+5|y|)^2+8A(x,y)}-(3|x|+5|y|)}{8}\right),\ \text{with } A(x,y)=3x^2+5y^2-7.$$

4. **Solution:** Given a linear normed space $(X,\|\cdot\|)$, define $\rho: X\times X\to\mathbb{R}$ by

$$\rho(x,y)=\|x-y\|,\ \forall x,y\in X.$$

It is easy to check that ρ is a metric. In fact, ρ satisfies

(1) $\rho(x,y)=\|x-y\|>0$ if $x\neq y$ and $\rho(x,x)=\|x-x\|=0$.
(2) $\rho(x,y)=\|x-y\|=\|(-1)(y-x)\|=|-1|\|y-x\|=\rho(y,x)$.
(3) $\rho(x,y)+\rho(y,z)=\|x-y\|+\|y-z\|\geq\|x-y+y-z\|=\|x-z\|=\rho(x,z)$.

Moreover, ρ satisfies

(i) $\rho(ax,ay)=\|ax-ay\|=|a|\|x-y\|=|a|\rho(x,y),\ \forall a\in\mathbb{F}$ (homogeneity)
(ii) $\rho(x+z,y+z)=\|x+z-(y+z)\|=\|x-y\|=\rho(x,y),\ \forall z\in X$ (translation invariance)

Conversely, if a metric ρ on a vector space X satisfies the properties

(i) $\rho(ax,ay)=|a|\rho(x,y),\ \forall a\in\mathbb{F}$ (homogeneity)
(ii) $\rho(x+z,y+z)=\rho(x,y),\ \forall z\in X$ (translation invariance)

then we could define a norm on X by

$$\|x\|=\rho(x,0),\quad\forall x\in X.$$

we will prove that $\|\cdot\|$ is a norm. In fact,

(1) $\|x\|=\rho(x,0)\geq 0$ for all $x\in X$, and the equality holds if and only if $x=0$.
(2) $\|ax\|=\rho(ax,0)=|a|\rho(x,0)=|a|\|x\|$ for all $x\in X,\ a\in\mathbb{F}$.
(3) $\|x+y\|=\rho(x+y,0)\leq\rho(x+y,y)+\rho(y,0)=\rho(x,0)+\rho(y,0)=\|x\|+\|y\|$ for all $x,y\in X$.

5. **Solution:** Let X be a non-empty set. Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x,y) = 1, \text{ if } x \neq y;$$

$$\rho(x,y) = 0, \text{ if } x = y.$$

It is easy to check that ρ is a metric. In fact, ρ satisfies

(1) $\rho(x,y) = 1 > 0$ if $x \neq y$ and $\rho(x,x) = 0$.

(2) $\rho(x,y) = \rho(y,x)$.

(3) For any $x, y, z \in X$,

* If x, y, z are all distinct, then $\rho(x,y) + \rho(y,z) = 1 + 1 \geq 1 = \rho(x,z)$;
* If exactly two of x, y, z are the same, then $\rho(x,y) + \rho(y,z) \geq 1 + 0 = 1 \geq \rho(x,z)$;
* If $x = y = z$, then we have $\rho(x,y) + \rho(y,z) = 0 + 0 = 0 = \rho(x,z)$.

In all, $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$ for all $x, y, z \in X$.

If a sequence $\{a_n\}_{n\in\mathbb{N}}$ in (X,ρ) is convergent, then there exists $N \in \mathbb{N}$ such that for $\forall\, \varepsilon \in (0,1)$, we have

$$\rho(a_n, a_m) < \varepsilon, \text{ whenever } n, m \geq N.$$

Since $\varepsilon < 1$, by the definition of the metric ρ, we have

$$a_n = a_m, \quad \forall n, m \geq N.$$

Hence there exists $l \in X$ such that $a_n = l$ for all $n > N$. Moreover, $l \in X$, so (X,ρ) is complete.